

HOMOGENEOUS MULTIPLICATIVE POLYNOMIAL LAWS ARE DETERMINANTS

F. VACCARINO

ABSTRACT

Let R be a ring and let B be a commutative ring. Let $p : R \rightarrow B$ be a homogeneous multiplicative polynomial law of degree n . The main result of this paper is to show that p is essentially a determinant, in the sense that p is obtained from a determinant by left and right composition with ring homomorphisms. This is achieved using results on the invariants of matrices in positive characteristic due to S. Donkin [3], [4], A. Zubkov [9], [10] and the author [7].

INTRODUCTION

Let R be a ring and B a commutative ring. Let $p : R \rightarrow B$ be a homogeneous multiplicative polynomial law of degree n . The definition of polynomial law will be recalled below and the reader can think about this as a polynomial map between rings that preserves the product and the identity. We want to understand the links between p and the usual determinant of $n \times n$ matrices, which is itself a homogeneous multiplicative polynomial law of degree n .

In order to state the main result of this paper we have to introduce some objects.

Let S be a set and let $F_S := \mathbb{Z}\langle x_s \rangle_{s \in S}$ be the free ring on it. Let $\mathbb{Z}[x_{i,j}^s]_{1 \leq i,j \leq n, s \in S}$ be the ring of polynomials in the variables $x_{i,j}^s$ with $1 \leq i, j \leq n$, $s \in S$ and let $M_n(\mathbb{Z}[x_{i,j}^s])$ denote the ring of $n \times n$ matrices with entries in it.

Let $j_n : F_S \rightarrow M_n(\mathbb{Z}[x_{i,j}^s])$ be given by $j_n(x_s) = \sum_{i,j=1}^n x_{i,j}^s e_{i,j}$, where $(e_{i,j})_{h,k} = \delta_{i,h} \delta_{j,k}$ for $i, j, h, k = 1, \dots, n$.

Let $E_S(n)$ be the subring of $\mathbb{Z}[x_{i,j}^s]_{1 \leq i,j \leq n, s \in S}$ generated by the coefficients of the characteristic polynomials of the elements of $j_n(F_S)$. Then $\det \cdot j_n$ maps F_S to $E_S(n)$.

Let $G = GL_n(\mathbb{Z})$ act by simultaneous conjugation on the direct sum of $\#S$ copies of $M_n(\mathbb{Z})$. Since $\mathbb{Z}[x_{i,j}^s]_{1 \leq i,j \leq n, s \in S}$ is the symmetric algebra of the direct sum of $\#S$ copies of $M_n(\mathbb{Z})$, then G acts on it as an automorphism group. We denote by $\mathbb{Z}[x_{i,j}^s]^G$ the ring

of invariants for this action. Then, clearly $\mathbb{Z}[x_{i,j}^s]^G$ contains $E_S(n)$. In fact, by works of S. Donkin and A. Zubkov ([3], [4], [9] and [10]), we have

$$\mathbb{Z}[x_{i,j}^s]^G = E_S(n).$$

We are now able to state the main result of this paper.

Theorem. *Let B be a commutative ring and let $p : F_S \rightarrow B$ be a homogeneous multiplicative polynomial law of degree n . There is a unique ring homomorphism $\phi : \mathbb{Z}[x_{i,j}^s]^G \rightarrow B$ such that the following diagram commutes:*

$$\begin{array}{ccc} F_S & \xrightarrow{p} & B \\ j_n \downarrow & & \uparrow \phi \\ j_n(F_S) & \xrightarrow{\det} & \mathbb{Z}[x_{i,j}^s]^G, \end{array}$$

i.e. p is a determinant composed with ring homomorphisms.

Let R be a ring and let

$$0 \rightarrow K \rightarrow F_S \xrightarrow{\pi} R \rightarrow 0$$

be any presentation of R by means of generators and relations.

Let B be a commutative ring and let $p : R \rightarrow B$ be a homogeneous multiplicative polynomial law of degree n . Now $\pi \cdot p : F_S \rightarrow B$ is a homogenous multiplicative polynomial law of degree n , by the previous theorem there is a unique $\rho : \mathbb{Z}[x_{i,j}^s]^G \rightarrow B$ such that

$$p(r) = (p \cdot \pi)(f) = (\rho \cdot \det \cdot j_n)(f),$$

for all $r \in R$, where $f \in F_S$ is such that $\pi(f) = r$, i.e. p is a determinant composed with ring homomorphisms.

The paper is divided into three sections: in the first one we recall some generalities on polynomial laws and divided powers algebras.

In the second section we recast some results due to S. Donkin, A. Zubkov and the author on the invariants of matrices in positive characteristic.

In the third section we give the main result.

I would like to thank C.Procesi and M.Brion for useful discussions.

§0 CONVENTIONS AND NOTATIONS

Except otherwise stated all the rings (algebras over a commutative ring) should be understood associative and with multiplicative identity.

We denote by \mathbb{N} the set of non-negative integers and by \mathbb{Z} the ring of integers.

Let R be a commutative ring and let n be any positive integer: we denote by $M_n(R)$ the ring of $n \times n$ matrices with entries in R .

Let A be a set, we denote by $\#A$ its cardinality.

For A a set and any additive monoid M , we denote by $M^{(A)}$ the set of functions $f : A \rightarrow M$ with finite support.

Let $\alpha \in M^{(A)}$, we denote by $|\alpha|$ the (finite) sum $\sum_{a \in A} \alpha(a)$.

Let S be a set. We put

- $F_S := \mathbb{Z}\langle x_s \rangle_{s \in S}$ for the free ring over S ;
- F_S^+ for the augmentation ideal of F_S (non-unital subring);
- $P_S := \mathbb{Z}[x_s]_{s \in S}$ for the free commutative ring over S .

§1 POLYNOMIAL LAWS AND DIVIDED POWERS

1.1 Let \mathbb{K} be a commutative ring. Let us recall the definition of a kind of map between \mathbb{K} -modules that generalizes the concept of polynomial map between free \mathbb{K} -modules (see [2] or [5]).

Definition 1.1.1. Let A and B be two \mathbb{K} -modules. A *polynomial law* φ from A to B is a family of mappings $\varphi_L : L \otimes_{\mathbb{K}} A \longrightarrow L \otimes_{\mathbb{K}} B$, with L varying in the family of commutative \mathbb{K} -algebras, such that the following diagram commutes:

$$\begin{array}{ccc} L \otimes_{\mathbb{K}} A & \xrightarrow{\varphi_L} & L \otimes_{\mathbb{K}} B \\ f \otimes 1_A \downarrow & & \downarrow f \otimes 1_B \\ M \otimes_{\mathbb{K}} A & \xrightarrow{\varphi_M} & M \otimes_{\mathbb{K}} B, \end{array}$$

for all L, M commutative \mathbb{K} -algebras and all homomorphisms of \mathbb{K} -algebras $f : L \longrightarrow M$.

Definition 1.1.2. Let $n \in \mathbb{N}$, if $\varphi_L(au) = a^n \varphi_L(u)$, for all $a \in L$, $u \in L \otimes_{\mathbb{K}} A$ and all commutative \mathbb{K} -algebras L , then φ will be said *homogeneous of degree n*.

Definition 1.1.3. If A and B are two \mathbb{K} -algebras and

$$\begin{cases} \varphi_L(xy) &= \varphi_L(x)\varphi_L(y) \\ \varphi_L(1_{L \otimes A}) &= 1_{L \otimes B}, \end{cases}$$

for all commutative \mathbb{K} -algebras L and for all $x, y \in L \otimes A$, then φ is called *multiplicative*.

Let A and B be two \mathbb{K} -modules and $\varphi : A \rightarrow B$ be a polynomial law. We recall the following result on polynomial laws, which is a restatement of Théorème I.1 of [5].

Proposition 1.1.4. *Let S be a set.*

- (1) *Let $L = \mathbb{K} \otimes P_S$ and let $\{a_s : s \in S\} \subset A$ be such that $a_s = 0$ except for a finite number of $s \in S$, then there exist $\varphi_\xi((a_s)) \in B$, with $\xi \in \mathbb{N}^{(S)}$, such that:*

$$\varphi_L \left(\sum_{s \in S} x_s \otimes a_s \right) = \sum_{\xi \in \mathbb{N}^{(S)}} x^\xi \otimes \varphi_\xi((a_s)),$$

where $x^\xi := \prod_{s \in S} x_s^{\xi_s}$.

- (2) *Let R be any commutative \mathbb{K} -algebra and let $(r_s)_{s \in S} \subset R$, then:*

$$\varphi_R \left(\sum_{s \in S} r_s \otimes a_s \right) = \sum_{\xi \in \mathbb{N}^{(S)}} r^\xi \otimes \varphi_\xi((a_s)),$$

where $r^\xi := \prod_{s \in S} r_s^{\xi_s}$.

- (3) *If φ is homogeneous of degree n , then in the previous sum one has $\varphi_\xi((a_s)) = 0$ if $|\xi|$ is different from n . That is:*

$$\varphi_R \left(\sum_{a \in A} r_a \otimes a \right) = \sum_{\xi \in \mathbb{N}^{(A)}, |\xi|=n} r^\xi \otimes \varphi_\xi((a)).$$

In particular, if φ is homogeneous of degree 0 or 1, then it is constant or linear, respectively.

Let S be a set, Proposition 1.1.4 means that a polynomial law $\varphi : A \rightarrow B$ is completely determined by its coefficients $\varphi_\xi((a_s))$, with $(a_s)_{s \in S} \in A^{(S)}$.

Remark 1.1.5. If A is a free \mathbb{K} -module and $\{a_t : t \in T\}$ is a basis of A , then φ is completely determined by its coefficients $\varphi_\xi((a_t))$, with $\xi \in \mathbb{N}^{(T)}$. If also B is a free \mathbb{K} -module with basis $\{b_u : u \in U\}$, then $\varphi_\xi((a_t)) = \sum_{u \in U} \lambda_u(\xi) b_u$. Let $a = \sum_{t \in T} \mu_t a_t \in A$. Since only a finite number of μ_t and $\lambda_u(\xi)$ are different from zero, the following makes sense:

$$\varphi(a) = \varphi \left(\sum_{t \in T} \mu_t a_t \right) = \sum_{\xi \in \mathbb{N}^{(T)}} \mu^\xi \varphi_\xi((a_t)) = \sum_{\xi \in \mathbb{N}^{(T)}} \mu^\xi \left(\sum_{u \in U} \lambda_u(\xi) b_u \right) = \sum_{u \in U} \left(\sum_{\xi \in \mathbb{N}^{(T)}} \lambda_u(\xi) \mu^\xi \right) b_u.$$

Hence, if both A and B are free \mathbb{K} -modules, a polynomial law between them is simply a polynomial map.

1.2 Let \mathbb{K} be any commutative ring with identity. For a \mathbb{K} -module M let $\Gamma(M)$ denote its divided powers algebra (see [2], [5]). This is a unital commutative \mathbb{K} -algebra,

with generators $m^{(k)}$, with $m \in M$, $k \in \mathbb{Z}$ and relations, for all $m, n \in M$:

- (i) $m^{(i)} = 0, \quad \forall i < 0;$
- (ii) $m^{(0)} = 1_{\mathbb{K}}, \quad \forall m \in M;$
- (iii) $(rm)^{(i)} = r^i m^{(i)}, \quad \forall r \in R, \forall i \in \mathbb{N};$
- (iv) $(m + n)^{(k)} = \sum_{i+j=k} m^{(i)} n^{(j)}, \quad \forall k \in \mathbb{N};$
- (v) $m^{(i)} m^{(j)} = \binom{i+j}{i} m^{(i+j)}, \quad \forall i, j \in \mathbb{N}.$

The \mathbb{K} -module $\Gamma(M)$ is generated by products (over arbitrary index sets I) $\prod_{i \in I} x_i^{(\alpha_i)}$ of the above generators, it is clear that $\prod_{i \in I} x_i^{(\alpha_i)} = 0$ if $\alpha_i < 0$ for some $i \in I$. The divided powers algebra $\Gamma(M)$ is a \mathbb{N} -graded algebra with homogeneous components $\Gamma_k := \Gamma_k(M)$, ($k \in \mathbb{N}$), the submodule generated by $\{\prod_{i \in I} x_i^{(\alpha_i)} : |\alpha| = k\}$. Note that $\Gamma_0 \cong \mathbb{K}$ and $\Gamma_1 \cong M$. Γ is a functor from \mathbb{K} -modules to commutative unital graded \mathbb{K} -algebras.

Indeed for any morphism of \mathbb{K} -modules $f : M \rightarrow N$ there exists a unique morphism of graded \mathbb{K} -algebras $\Gamma(f) : \Gamma(M) \rightarrow \Gamma(N)$ such that $\Gamma(f)(x^{(n)}) = f(x)^{(n)}$, for any $x \in M$ and $n \geq 0$. From this it follows easily that Γ is exact.

Furthermore $\Gamma(L \otimes_{\mathbb{K}} M) \cong L \otimes_{\mathbb{K}} \Gamma(M)$ as graded rings by means of $(1 \otimes x)^{(n)} \mapsto 1 \otimes x^{(n)}$. Thus the map $\Gamma(f)$ commutes with extensions of scalars.

If A is a (unital) \mathbb{K} -algebra, then $\Gamma_k(A)$ is a (unital) \mathbb{K} -algebra too (see [6]). To distinguish the new multiplication on $\Gamma_k(A)$ from the one of $\Gamma(A)$, we denote it by “ τ_k ”. We have:

$$\begin{aligned} \prod_{i \in I} a_i^{(\alpha_i)} \tau_k \prod_{j \in J} b_j^{(\beta_j)} &:= (\prod_{i \in I} a_i^{(\alpha_i)}) \tau_k (\prod_{j \in J} b_j^{(\beta_j)}) \\ &:= \sum_{(\lambda_{ij}) \in M(\alpha, \beta)} \prod_{(i,j) \in I \times J} (a_i b_j)^{(\lambda_{ij})}, \end{aligned}$$

where $M(\alpha, \beta) := \{(\lambda_{ij}) \in \mathbb{N}^{(I \times J)} : \sum_{i \in I} \lambda_{ij} = \beta_j, \forall j \in J; \sum_{j \in J} \lambda_{ij} = \alpha_i, \forall i \in I\}$ and $\prod_{i \in I} a_i^{(\alpha_i)}, \prod_{j \in J} b_j^{(\beta_j)} \in \Gamma_k(A)$.

Let us denote by $\gamma_n := (\gamma_{n,L})$ the polynomial law given by the composition $L \otimes M \rightarrow \Gamma_n(L \otimes M) \rightarrow L \otimes \Gamma_n(M)$, then γ_n is homogeneous of degree n .

There is a property proved by Roby in [6], which motivates our introduction of divided powers.

Theorem 1.2.1. *Let A and B be two \mathbb{K} -algebras. The set of homogeneous multiplicative polynomial laws of degree n from A to B is in bijection with the set of all homomorphisms of \mathbb{K} -algebras from $\Gamma_n(A)$ to B . Namely, given any homogeneous multiplicative polynomial law $f : A \rightarrow B$ of degree n , there exists a unique homomorphism of \mathbb{K} -algebras $\phi : \Gamma_n(A) \rightarrow B$ such that $f_L = (1_L \otimes \phi) \cdot \gamma_{n,L}$, for any commutative \mathbb{K} -algebra L .*

1.3 There is another result that we need to recall.

Let B be a commutative ring and let $M_n(B)$ be the ring of $n \times n$ matrices over B . Let $b \in M_n(B)$ and denote by $e_i(b)$ the i -th coefficient of the characteristic polynomial of b , i.e. the trace of $\wedge^i(b)$.

Let R be a ring, we denote by $(R)^{ab}$ its abelianization, that is, its quotient by the ideal generated by the commutators of its elements.

The following can be found in [8].

Proposition 1.3.1. *The ring $\Gamma_n(M_n(B))^{ab}$ is isomorphic to B . The canonical projection $\alpha_n := \alpha_n(M_n(B)) : \Gamma_n(M_n(B)) \rightarrow \Gamma_n(M_n(B))^{ab}$ is such that, for all $b \in M_n(B)$ and $0 \leq i \leq n$,*

$$\alpha_n(1^{(n-i)} b^{(i)}) = e_i(b).$$

For further readings on these topics we refer to [2], [5], [6], [8].

§2 DIVIDED POWERS AND INVARIANTS OF MATRICES

2.1 Let us introduce the following notation: let S be a set, we put

$A_S(n) := \mathbb{Z}[x_{ij}^s]_{1 \leq i,j \leq n, s \in S}$, the symmetric algebra on the direct sum of $\#S$ copies of $M_n(\mathbb{Z})$;

$B_S(n) := M_n(A_S(n))$;

$G := Gl_n(\mathbb{Z})$;

$C_S(n) := A_S(n)^G$, the ring of polynomial invariants with respect to the action of G on $A_S(n)$ induced by simultaneous conjugation on the direct sum of $\#S$ copies of $M_n(\mathbb{Z})$.

2.2 Let $j_n : F_S \rightarrow B_S(n)$ be given by $j_n(x_s) = \zeta_s := \sum_{i,j=1}^n x_{i,j}^s e_{i,j}$, where $(e_{i,j})_{h,k} = \delta_{i,h} \delta_{j,k}$ for $i, j, h, k = 1, \dots, n$. The ζ_s are the so-called "generic matrices of order n ". Let $\beta_n : \Gamma_n(B_S(n)) \rightarrow \Gamma_n(B_S(n))^{ab}$ be the canonical projection, then $A_S(n) \cong \Gamma_n(B_S(n))^{ab}$ by Prop.1.3.1. We set $E_S(n) := \beta_n(\Gamma_n(j_n(F_S))) \hookrightarrow A_S(n)$, by Prop.1.3.1 it is the subring of $A_S(n)$ generated by the $e_i(j_n(f))$, where $i \in \mathbb{N}$ and $f \in F_S$. By the exactness properties of Γ_n and $(-)^{ab}$ there exists a unique ring epimorphism $\pi_n : \Gamma_n(F_S)^{ab} \rightarrow E_S(n)$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma_n(F_S) & \xrightarrow{\Gamma_n(j_n)} & \Gamma_n(j_n(F_S)) \\ \alpha_n \downarrow & & \downarrow \beta_n \\ \Gamma_n(F_S)^{ab} & \xrightarrow{\pi_n} & E_S(n). \end{array}$$

Remark 2.2.1. By the previous discussion the polynomial law $j_n(F_S) \rightarrow E_S(n)$ that corresponds to β_n , via theorem 1.2.1, is the restriction to $j_n(F_S)$ of the determinant.

Recall that $E_S(n)$ is a subring of $C_S(n)$.

The Donkin-Zubkov Theorem on invariant of matrices can be stated in the following way , see [3], [4], [9] and [10]. (This Theorem was firstly proved by S. Donkin and then by A. Zubkov in another way).

Theorem (Donkin-Zubkov). *The ring $C_S(n)$ of polynomial invariants of the direct sum of $\#S$ copies of $n \times n$ matrices is equal to $E_S(n)$.*

Then we have a surjection $\pi_n : \Gamma_n(F_S)^{ab} \rightarrow E_S(n) = C_S(n)$.

The following result is proved in [7].

Theorem 2.2.2. *The map $\pi_n : \Gamma_n(F_S)^{ab} \rightarrow C_S(n)$ is an isomorphism of graded rings.*

The proof of the theorem goes as follows: one sees that the natural multidegree of F_S induces a multidegree on $\Gamma_n(F_S)^{ab}$ and the following is showed to be an homomorphism of graded rings for all n :

$$\rho_n : \begin{cases} \Gamma_n(F_S)^{ab} \rightarrow \Gamma_{n-1}(F_S)^{ab} \\ 1^{(n-|\alpha|)} \prod_{i \in I} a_i^{(\alpha_i)} \mapsto 1^{(n-1-|\alpha|)} \prod_{i \in I} a_i^{(\alpha_i)}, \end{cases}$$

where $1^{(n-|\alpha|)} \prod_{i \in I} a_i^{(\alpha_i)} \in \Gamma_n(F_S)^{ab}$.

Let $\delta_n : A_S(n) \rightarrow A_S(n-1)$ be the natural projection and denote again by δ_n its restriction to $C_S(n)$. The following diagram is showed to be a commutative diagram in the category of graded rings:

$$\begin{array}{ccc} \Gamma_n(F_S)^{ab} & \xrightarrow{\rho_n} & \Gamma_{n-1}(F_S)^{ab} \\ \pi_n \downarrow & & \downarrow \pi_{n-1} \\ C_S(n) & \xrightarrow{\delta_n} & C_S(n-1). \end{array}$$

Take the graded inverse limit (with respect to n) of $(\Gamma_n(F_S)^{ab}, \rho_n)$. This is showed to be isomorphic, via the previous diagram, to the graded inverse limit (with respect to δ_n) of the rings $C_S(n)$.

S. Donkin showed that the latter is the free commutative ring $\bigotimes_{\mu \in \psi} \Lambda_\mu$, where each Λ_μ is a copy of the ring of symmetric functions and ψ is the set of (equivalence classes with respect to cyclic permutations of) primitive monomials (see [3],[4]).

Let $\sigma_n : \bigotimes_{\mu \in \psi} \Lambda_\mu \rightarrow \Gamma_n(F_S)^{ab}$ be the canonical projection from $\bigotimes_{\mu \in \psi} \Lambda_\mu$ considered as the graded inverse limit of the $\Gamma_n(F_S)^{ab}$.

The proof goes on showing that the following sequence is exact:

$$0 \rightarrow \langle \{e_{n+1+k}(f) : k \in \mathbb{N} \text{ and } f \in F_S^+\} \rangle \rightarrow \bigotimes_{\mu \in \psi} \Lambda_\mu \xrightarrow{\sigma_n} \Gamma_n(F_S)^{ab} \rightarrow 0.$$

Note that it makes sense to write $e_i(f)$ since this can be expressed as a polynomial in the $e_j(m)$, with m a monomial, by means of Amitsur's Formula (see cite1).

In [10] the following is proved.

Theorem (Zubkov). *The following sequence is exact:*

$$0 \rightarrow \langle \{e_{n+1+k}(f) : k \in \mathbb{N} \text{ and } f \in F_S^+\} \rangle \rightarrow \bigotimes_{\mu \in \psi} \Lambda_\mu \xrightarrow{\theta_n} C_S(n) \rightarrow 0,$$

where θ_n is the canonical projection from $\bigotimes_{\mu \in \psi} \Lambda_\mu$ considered as the graded inverse limit of the $C_S(n)$.

The result then follows by comparing the presentation of $\Gamma_n(F_S)^{ab}$ with the one of $C_S(n)$ by means of π_n and its limit.

Corollary 2.2.3. *Via Theorem 1.2.1, the ring isomorphism $\Gamma_n(F_S)^{ab} \cong C_S(n)$ corresponds to the polynomial law $\det \cdot j_n : F_S \rightarrow C_S(n)$.*

Proof. It follows from Th.2.2.2 and Remark 2.2.1. \square

§3 PROOF OF THE MAIN RESULT

Let $p : F_S \rightarrow B$ be a homogeneous multiplicative polynomial law of degree n , then, by Theorem 1.2.1 there exists a unique ring homomorphism $\phi : \Gamma_n(F_S)^{ab} \rightarrow B$ such that $p_L = (1_L \otimes \phi) \cdot \gamma_{n,L}$ for any commutative ring L . But $\Gamma_n(F_S)^{ab} \cong C_S(n)$ by Theorem 2.2.2 and Corollary 2.2.3 gives the arrows making the following diagram to commute:

$$\begin{array}{ccc} F_S & \xrightarrow{p} & B \\ j_n \downarrow & & \uparrow \phi \\ j_n(F_S) & \xrightarrow{\det} & C_S(n). \end{array}$$

REFERENCES

- [1] S.A.Amitsur, *On the Characteristic Polynomial of a Sum of Matrices*, Linear and Multilinear Algebra **8** (1980), 177-182.
- [2] P.Berthelot , A.Ogus, *Notes on Crystalline Cohomology*, Princeton University Press, Princeton N.J. (1978).
- [3] S.Donkin, *Invariant functions on matrices*, Math. Proc. Camb. Phil. Soc. **113** (1993), 23-43.
- [4] S.Donkin, *Invariants of several matrices*, Invent. math **110** (1992), 389-401.
- [5] N.Roby, *Lois polynômes et lois formelles en théorie des modules*, Ann.Ecole Norm. Sup. **80** (1963), 213-348.
- [6] N.Roby, *Lois polynômes multiplicatives universelles*, C.R.Acad.SC.Paris **290** (1980), 869-871.
- [7] F.Vaccarino, *On the Invariants of Matrices and the Embedding Problem*, Preprint (1996).
- [8] D.Ziplies, *A Characterization of the Norm of an Azumaya Algebra of Constat Rank through the Divided Powers Algebra of an Algebra*, Beitraege Algebra Geom. **22** (1986), 53-70.
- [9] A.N.Zubkov, *Endomorphisms of Tensor Products of Exterior Powers and Procesi Hypothesis*, Comm.Algebra **22** (1994), 6385-6399.
- [10] A.N.Zubkov, *Adjoint action invariants of classical groups*, Algebra and Logic **38**, n.5 (1999), 549-584.